

*Duality Transformation
of
non-Abelian Tensor Gauge Fields*

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Abstract

For non-Abelian tensor gauge fields we have found an alternative form of duality transformation, which has the property that the direct and the inverse transformations coincide. This duality transformation has the desired property that the direct and the inverse transformations map Lagrangian forms into each other.

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1 Introduction

In the recent decades string field theories and higher-spin field theories became a subject of intensive research. One of the purposes of this development is to find out an effective method for calculating *off-shell* scattering amplitudes of high spin fields.

In the field theoretical approach the Lagrangian and S-matrix formulations of *free* massless Abelian tensor gauge fields have been constructed in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The *interaction* of higher-spin fields has been studied in the light-cone formalism and in the covariant formulation of the theories [12, 13, 14, 15, 16, 17, 18, 19, 20]. In string field theory the interaction of higher-spin fields has been studied in [24, 25, 26, 27, 28, 29, 30, 31, 32]. The interacting field theories in anti-de Sitter space-time background are reviewed in [21, 22, 23].

In general the concept of local gauge invariance allows one to define the non-Abelian gauge fields [33], to derive their dynamical field equations and to develop a universal point of view on matter interactions as resulting from the exchange of spin-one gauge quanta. A possible extension of the gauge principle which defines the interaction of high-spin gauge fields has been made recently in [34]. The resulting gauge invariant Lagrangian defines cubic and quartic self-interactions of charged gauge quanta carrying a spin larger than one [34, 35, 36].

Recall that in these publications it was found that there exists not one but a pair, δ and $\tilde{\delta}$, of complementary non-Abelian gauge transformations acting on tensor gauge fields of the rank $s+1$: $A_{\mu\lambda_1\dots\lambda_s}^a$. These are totally symmetric tensors with respect to the indices $\lambda_1\dots\lambda_s$, but *a priori* have no symmetries with respect to the first index μ . The *extended gauge transformation* δ_ξ has the following form [34, 35, 36]:

$$\begin{aligned}\delta_\xi A_\mu^a &= (\delta^{ab}\partial_\mu + gf^{acb}A_\mu^c)\xi^b, \\ \delta_\xi A_{\mu\lambda_1}^a &= (\delta^{ab}\partial_\mu + gf^{acb}A_\mu^c)\xi_{\lambda_1}^b + gf^{acb}A_{\mu\lambda_1}^c\xi^b, \\ \delta_\xi A_{\mu\lambda_1\lambda_2}^a &= (\delta^{ab}\partial_\mu + gf^{acb}A_\mu^c)\xi_{\lambda_1\lambda_2}^b + gf^{acb}(A_{\mu\lambda_1}^c\xi_{\lambda_2}^b + A_{\mu\lambda_2}^c\xi_{\lambda_1}^b + A_{\mu\lambda_1\lambda_2}^c\xi^b), \\ &\dots\dots\dots\end{aligned}\tag{1}$$

and the *complementary gauge transformation* $\tilde{\delta}_\eta$ is [36] :

$$\begin{aligned}\tilde{\delta}_\eta A_\mu^a &= (\delta^{ab}\partial_\mu + gf^{acb}A_\mu^c)\eta^b, \\ \tilde{\delta}_\eta A_{\mu\lambda_1}^a &= (\delta^{ab}\partial_{\lambda_1} + gf^{acb}A_{\lambda_1}^c)\eta_\mu^b + gf^{acb}A_{\mu\lambda_1}^c\eta^b, \\ \tilde{\delta}_\eta A_{\mu\lambda_1\lambda_2}^a &= (\delta^{ab}\partial_{\lambda_1} + gf^{acb}A_{\lambda_1}^c)\eta_{\mu\lambda_2}^b + (\delta^{ab}\partial_{\lambda_2} + gf^{acb}A_{\lambda_2}^c)\eta_{\mu\lambda_1}^b + \\ &\quad + gf^{acb}(A_{\mu\lambda_1}^c\eta_{\lambda_2}^b + A_{\mu\lambda_2}^c\eta_{\lambda_1}^b + A_{\lambda_1\lambda_2}^c\eta_\mu^b + A_{\lambda_2\lambda_1}^c\eta_\mu^b + A_{\mu\lambda_1\lambda_2}^c\eta^b),\end{aligned}\tag{2}$$

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These transformations form a closed algebraic structure in the sense that

$$[\delta_\eta, \delta_\xi] A_{\mu\lambda_1\lambda_2\dots\lambda_s} = -ig \delta_\xi A_{\mu\lambda_1\lambda_2\dots\lambda_s}, \quad [\tilde{\delta}_\eta, \tilde{\delta}_\xi] A_{\mu\lambda_1\lambda_2\dots\lambda_s} = -ig \tilde{\delta}_\xi A_{\mu\lambda_1\lambda_2\dots\lambda_s}$$

and have the same composition law for the gauge parameters:

$$\begin{aligned} \zeta &= [\eta, \xi] \\ \zeta_{\lambda_1} &= [\eta, \xi_{\lambda_1}] + [\eta_{\lambda_1}, \xi] \\ \zeta_{\lambda_1\lambda_2} &= [\eta, \xi_{\lambda_1\lambda_2}] + [\eta_{\lambda_1}, \xi_{\lambda_2}] + [\eta_{\lambda_2}, \xi_{\lambda_1}] + [\eta_{\lambda_1\lambda_2}, \xi], \\ &\dots . \dots \end{aligned} \tag{3}$$

The transformations δ_ξ and $\tilde{\delta}_\eta$ do not coincide and are *complementary* to each other in the following sense: in δ_ξ the derivatives of the gauge parameters $\{\xi\}$ are over the first index μ , while in $\tilde{\delta}_\eta$ the derivatives of the gauge parameters $\{\eta\}$ are over the rest of the totally symmetric indices $\lambda_1\dots\lambda_s$, so that together they cover all indices of the nonsymmetric tensor gauge fields $A_{\mu\lambda_1\dots\lambda_s}^a$ (recall that these tensor gauge fields are not symmetric with respect to the index μ and the rest of the indices $\lambda_1\dots\lambda_s$). Therefore the above transformations (1) and (2) are complementary representations of the same infinite-dimensional gauge group \mathcal{G} with the associative algebra (3) [36].

The generalized field strength tensors $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$ transform *homogeneously* with respect to the transformations δ_ξ (1) [34, 35] and allow to construct the gauge invariant Lagrangian $\mathcal{L}(A)$ which describes dynamical tensor gauge bosons of all ranks [34, 35, 36, 37]. In recent publication [38] the authors constructed complementary field strength tensors $\tilde{G}_{\mu\nu,\lambda_1\dots\lambda_s}^a$ which are transforming homogeneously,³ now with respect to the $\tilde{\delta}_\eta$ (2) [38] and allow to construct the corresponding gauge invariant Lagrangian $\tilde{\mathcal{L}}(A)$.

Thus there are two Lagrangian forms $\mathcal{L}(A)$ and $\tilde{\mathcal{L}}(A)$ for the same tensor gauge fields $A_{\mu\lambda_1\dots\lambda_s}^a$ which are fully invariant with respect to the corresponding gauge transformations (1) and (2) [38]

$$\delta_\xi \mathcal{L}(A) = 0, \quad \tilde{\delta}_\eta \tilde{\mathcal{L}}(A) = 0.$$

The natural question which was raised at this point was to find out a possible relation between these Lagrangian forms. It has been found that the following duality

³The field strength tensors $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$ and $\tilde{G}_{\mu\nu,\lambda_1\dots\lambda_s}^a$ are antisymmetric in their first two indices and are totally symmetric with respect to the rest of the indices. The explicit form of these tensors is given in [34, 35, 38].

transformation[38]

$$\begin{aligned}
\tilde{A}_{\mu\lambda_1} &= A_{\lambda_1\mu}, \\
\tilde{A}_{\mu\lambda_1\lambda_2} &= \frac{1}{2}(A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}) - \frac{1}{2}A_{\mu\lambda_1\lambda_2}, \\
\tilde{A}_{\mu\lambda_1\lambda_2\lambda_3} &= \frac{1}{3}(A_{\lambda_1\mu\lambda_2\lambda_3} + A_{\lambda_2\mu\lambda_1\lambda_3} + A_{\lambda_3\mu\lambda_1\lambda_2}) - \frac{2}{3}A_{\mu\lambda_1\lambda_2\lambda_3}, \\
&\dots\dots\dots
\end{aligned} \tag{4}$$

maps the Lagrangian $\tilde{\mathcal{L}}(A)$ into the Lagrangian $\mathcal{L}(\tilde{A})$. This takes place because [38]

$$\tilde{G}_{\mu\nu,\lambda_1\dots\lambda_s}(A) = G_{\mu\nu,\lambda_1\dots\lambda_s}(\tilde{A})$$

and therefore

$$\tilde{\mathcal{L}}(A) = \mathcal{L}(\tilde{A}).$$

One can find also the inverse duality transformation [38]

$$\begin{aligned}
A_{\mu\lambda_1} &= \tilde{A}_{\lambda_1\mu}, \\
A_{\mu\lambda_1\lambda_2} &= \tilde{A}_{\lambda_1\mu\lambda_2} + \tilde{A}_{\lambda_2\mu\lambda_1}, \\
A_{\mu\lambda_1\lambda_2\lambda_3} &= \frac{1}{3}(\tilde{A}_{\lambda_1\mu\lambda_2\lambda_3} + \tilde{A}_{\lambda_2\mu\lambda_1\lambda_3} + \tilde{A}_{\lambda_3\mu\lambda_1\lambda_2}), \\
&\dots\dots\dots
\end{aligned} \tag{5}$$

The duality map (4) is one-to-one. The inverse transformation (5) has the following unusual property. If one applies the inverse transformation (5) now to the Lagrangian form $\mathcal{L}(A)$, one can see that the resulting expression can not be identified with the Lagrangian form $\tilde{\mathcal{L}}(\tilde{A})$, that is,

$$\mathcal{L}(A) \not\rightarrow \tilde{\mathcal{L}}(\tilde{A}).$$

In this article we would like to find out an explanation for this phenomenon. As we shall demonstrate there exists an infinite family (11) of duality transformations between tensor gauge fields. Within this family of duality transformations there is a unique one which has the property that the direct and the inverse transformations coincide. This duality transformation (12), (13), (14) has the desired property that the direct and the inverse transformations map \mathcal{L} to $\tilde{\mathcal{L}}$ and via versa

$$\mathcal{L} \leftrightarrow \tilde{\mathcal{L}}.$$

2 Two-Parameter Family of Duality Transformations

The general form of the duality transformation (4) is [38]

$$\tilde{A}_{\mu\lambda_1\dots\lambda_s} = \frac{1}{s}(A_{\lambda_1\mu\dots\lambda_s} + \dots + A_{\lambda_s\mu\dots\lambda_{s-1}}) - \frac{s-1}{s}A_{\mu\lambda_1\dots\lambda_s} \quad s = 1, 2, \dots \tag{6}$$

and can be expressed in the matrix form

$$\tilde{A}_{\mu\lambda_1\ldots\lambda_s} = M_{\mu\lambda_1\ldots\lambda_s}{}^{\nu\rho_1\ldots\rho_s} A_{\nu\rho_1\ldots\rho_s}, \quad (7)$$

where the matrix M and its inverse have the following structure:

$$M = \frac{1}{s}P - \frac{s-1}{s}\mathbb{I}, \quad M^{-1} = P. \quad (8)$$

Here we have two operators, the permutation operator P and the identity operator \mathbb{I} . The permutation operator P interchanges the index μ with the indices λ_i and sums the result over all $i = 1, 2, \dots, s$. We see that the inverse transformation M^{-1} does not coincide with the direct transformation M . This is the main obstacle preventing the inverse duality map to relate the Lagrangian form \mathcal{L} with $\tilde{\mathcal{L}}$, that is, $\mathcal{L}(A) \not\leftrightarrow \tilde{\mathcal{L}}(\tilde{A})$.

Let us consider the properties of the matrix M in more details. The permutation matrix P has the property

$$P^2 = (s-1)P + s\mathbb{I} \quad (9)$$

and therefore from (8) we can get the square of the duality matrix M

$$M^2 = \frac{1-s}{s}M + \frac{1}{s}\mathbb{I}. \quad (10)$$

From this relation we can clearly see that the square of the duality matrix M is not equal to one: $M^2 \neq \mathbb{I}$. We would like to find out an alternative duality map T for which $T^2 = \mathbb{I}$ and therefore $T^{-1} = T$.

The duality transformation (7), (8) is a linear transformation of the basic tensor gauge fields and can be defined by any nonsingular matrix T :

$$\tilde{A}_{\mu\lambda_1\ldots\lambda_s} = T_{\mu\lambda_1\ldots\lambda_s}{}^{\nu\rho_1\ldots\rho_s} A_{\nu\rho_1\ldots\rho_s}. \quad (11)$$

Let us consider a two (a, b) -parameter class of maps similar to (8)

$$T = aP + b\mathbb{I}.$$

Calculating the square of the matrix T and using the relation (9) we get

$$T^2 = (2ab + a^2(s-1))P + (b^2 + a^2s)\mathbb{I}.$$

Requiring that it is equal to the identity matrix we shall get a system of algebraic equations:

$$2ab + a^2(s-1) = 0, \quad b^2 + a^2s = 1.$$

The nontrivial solution gives the desired solution for T :

$$T = \frac{2}{s+1}P - \frac{s-1}{s+1}\mathbb{1}. \quad (12)$$

This matrix has the property that $T^2 = 1$ and therefore $T^{-1} = T$. Thus the transformation (11) will take the form

$$\tilde{A}_{\mu\lambda_1\dots\lambda_s} = \frac{2}{s+1}(A_{\lambda_1\mu\dots\lambda_s} + \dots + A_{\lambda_s\mu\dots\lambda_{s-1}}) - \frac{s-1}{s+1}A_{\mu\lambda_1\dots\lambda_s} \quad s = 1, 2, \dots \quad (13)$$

In particular, for the first values of s , we have

$$\begin{aligned} \tilde{A}_{\mu\lambda_1} &= A_{\lambda_1\mu}, \\ \tilde{A}_{\mu\lambda_1\lambda_2} &= \frac{2}{3}(A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}) - \frac{1}{3}A_{\mu\lambda_1\lambda_2}, \\ \tilde{A}_{\mu\lambda_1\lambda_2\lambda_3} &= \frac{1}{2}(A_{\lambda_1\mu\lambda_2\lambda_3} + A_{\lambda_2\mu\lambda_1\lambda_3} + A_{\lambda_3\mu\lambda_1\lambda_2}) - \frac{1}{2}A_{\mu\lambda_1\lambda_2\lambda_3}, \\ &\dots\dots\dots \end{aligned} \quad (14)$$

A posteriori one can get convinced that this duality map and its inverse coincide. This is the main difference between duality maps (4),(6) and (13),(14).

In (14) the first line defines the ordinary *transposition* and the subsequent lines define natural generalization of the transposition operation to the higher-dimensional tensors. In the next section we shall demonstrate that there is an infinite family of complementary gauge transformations $\tilde{\delta}_\eta$ which have the same structure as the complementary gauge transformation (2) and that the above transformation (12), (13), (14) defines a natural duality map between them.

3 Complementary Gauge Transformations

The observation made in [38], that the complementary gauge transformation (2) acting on the tensor gauge field \tilde{A} is identical with the extended gauge transformation (1), implies that the duality map serves as a similarity transformation between two representations of the same gauge algebra (3) (see also comment after formula (3)). Therefore requiring that the gauge field $\tilde{\mathcal{A}}(e)$ [36] transforms by the extended gauge transformation (1) we can find out the complementary gauge transformations of the tensor gauge field $\mathcal{A}(e)$ in the following form⁴:

$$\tilde{\delta}_\eta \mathcal{A}_\mu = T_\mu{}^\nu \partial_\nu \eta - ig T_\mu{}^\rho [T_\rho{}^\nu \mathcal{A}_\nu, \eta]. \quad (15)$$

To operate with this general formula one should expand the field $\mathcal{A}(e)$ as in [36] and use the explicit form of the matrices T . The explicit form of these matrices can be obtained

⁴The symmetric indices $\lambda_1, \dots, \lambda_s; \rho_1 \dots \rho_s$ are suppressed in this formula $T_{\mu\lambda_1\dots\lambda_s}{}^{\nu\rho_1\dots\rho_s} \sim T_\mu{}^\nu [36]$.

from (12), (13), (14)

$$\begin{aligned}
T_{\mu\lambda_1}^{\nu\rho_1} &= \delta_\mu^{\rho_1} \delta_{\lambda_1}^\nu, \\
T_{\mu\lambda_1\lambda_2}^{\nu\rho_1\rho_2} &= \frac{2}{3}(\delta_\mu^{\rho_1} \delta_{\lambda_1}^\nu \delta_{\lambda_2}^{\rho_2} + \delta_\mu^{\rho_2} \delta_{\lambda_1}^{\rho_1} \delta_{\lambda_2}^\nu) - \frac{1}{3} \delta_\mu^\nu \delta_{\lambda_1}^{\rho_1} \delta_{\lambda_2}^{\rho_2}, \\
T_{\mu\lambda_1\lambda_2\lambda_3}^{\nu\rho_1\rho_2\rho_3} &= \frac{1}{2}(\delta_\mu^{\rho_1} \delta_{\lambda_1}^\nu \delta_{\lambda_2}^{\rho_2} \delta_{\lambda_3}^{\rho_3} + \delta_\mu^{\rho_2} \delta_{\lambda_1}^{\rho_1} \delta_{\lambda_2}^\nu \delta_{\lambda_3}^{\rho_3} + \delta_\mu^{\rho_3} \delta_{\lambda_1}^{\rho_1} \delta_{\lambda_2}^{\rho_2} \delta_{\lambda_3}^\nu) - \frac{1}{3} \delta_\mu^\nu \delta_{\lambda_1}^{\rho_1} \delta_{\lambda_2}^{\rho_2} \delta_{\lambda_3}^{\rho_3}, \\
&\dots\dots\dots \dots\dots\dots
\end{aligned} \tag{16}$$

Thus the complementary gauge transformation $\tilde{\delta}_\eta$ of the tensor gauge fields is

$$\begin{aligned}
\tilde{\delta}_\eta A_\mu &= \partial_\mu \eta - ig[A_\mu, \eta], \\
\tilde{\delta}_\eta A_{\mu\lambda_1} &= \partial_{\lambda_1} \eta_\mu - ig[A_{\lambda_1}, \eta_\mu] - ig[A_{\mu\lambda_1}, \eta], \\
\tilde{\delta}_\eta A_{\mu\lambda_1\lambda_2} &= \frac{2}{3}(\partial_{\lambda_1} \eta_{\mu\lambda_2} - ig[A_{\lambda_1}, \eta_{\mu\lambda_2}] + \partial_{\lambda_2} \eta_{\mu\lambda_1} - ig[A_{\lambda_2}, \eta_{\mu\lambda_1}]) - \frac{1}{3}(\partial_\mu \eta_{\lambda_1\lambda_2} - ig[A_\mu, \eta_{\lambda_1\lambda_2}]) \\
&\quad - ig\frac{2}{3}[A_{\mu\lambda_1}, \eta_{\lambda_2}] - ig\frac{2}{3}[A_{\lambda_1\lambda_2}, \eta_\mu] + ig\frac{1}{3}[A_{\lambda_1\mu}, \eta_{\lambda_2}] \\
&\quad - ig\frac{2}{3}[A_{\mu\lambda_2}, \eta_{\lambda_1}] - ig\frac{2}{3}[A_{\lambda_2\lambda_1}, \eta_\mu] + ig\frac{1}{3}[A_{\lambda_2\mu}, \eta_{\lambda_1}] - ig[A_{\mu\lambda_1\lambda_2}, \eta], \\
&\dots\dots\dots \dots\dots\dots
\end{aligned} \tag{17}$$

where we have used the matrix notation $A_{\mu\lambda_1\dots\lambda_s} = A_{\mu\lambda_1\dots\lambda_s}^a L^a$ [34, 35, 36]. It is instructive to compare this complementary gauge transformation with (1) and (2). At zero coupling constant, $g = 0$, it gives

$$\begin{aligned}
\tilde{\delta}_\eta A_\mu &= \partial_\mu \eta, \\
\tilde{\delta}_\eta A_{\mu\lambda_1} &= \partial_{\lambda_1} \eta_\mu, \\
\tilde{\delta}_\eta A_{\mu\lambda_1\lambda_2} &= \frac{2}{3}(\partial_{\lambda_1} \eta_{\mu\lambda_2} + \partial_{\lambda_2} \eta_{\mu\lambda_1}) - \frac{1}{3} \partial_\mu \eta_{\lambda_1\lambda_2}, \\
&\dots\dots\dots \dots\dots\dots
\end{aligned} \tag{18}$$

and defines the behavior of the longitudinal parts of the tensor gauge field with respect to the symmetric indices $\lambda_1, \dots, \lambda_s$, in a way similar to (1) and (2). The corresponding covariant field strength tensors can also be constructed with the use of the matrix T :

$$\begin{aligned}
\tilde{G}_{\mu\nu} &\equiv G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \\
\tilde{G}_{\mu\nu, \lambda_1} &= \partial_\mu A_{\lambda_1\nu} - \partial_\nu A_{\lambda_1\mu} - ig[A_\mu, A_{\lambda_1\nu}] - ig[A_{\lambda_1\mu}, A_\nu], \\
\tilde{G}_{\mu\nu, \lambda_1\lambda_2} &= \partial_\mu \left(\frac{2}{3} A_{\lambda_1\nu\lambda_2} + \frac{2}{3} A_{\lambda_2\nu\lambda_1} - \frac{1}{3} A_{\nu\lambda_1\lambda_2} \right) - ig[A_\mu, \left(\frac{2}{3} A_{\lambda_1\nu\lambda_2} + \frac{2}{3} A_{\lambda_2\nu\lambda_1} - \frac{1}{3} A_{\nu\lambda_1\lambda_2} \right)] - \\
&\quad - \partial_\nu \left(\frac{2}{3} A_{\lambda_1\mu\lambda_2} + \frac{2}{3} A_{\lambda_2\mu\lambda_1} - \frac{1}{3} A_{\mu\lambda_1\lambda_2} \right) - ig \left[\left(\frac{2}{3} A_{\lambda_1\mu\lambda_2} + \frac{2}{3} A_{\lambda_2\mu\lambda_1} - \frac{1}{3} A_{\mu\lambda_1\lambda_2} \right), A_\nu \right] \\
&\quad - ig[A_{\lambda_1\mu}, A_{\lambda_2\nu}] - ig[A_{\lambda_2\mu}, A_{\lambda_1\nu}], \\
&\dots\dots\dots \dots\dots\dots
\end{aligned} \tag{19}$$

These field strength tensors transform homogeneously and allow to construct the invariant Lagrangian $\tilde{\mathcal{L}}(A)$ quadratic in field strength tensors [34, 35, 36, 38]. The dual transformation (13), (14) tells us now that

$$\tilde{G}_{\mu\nu,\lambda_1\ldots\lambda_s}(A) = G_{\mu\nu,\lambda_1\ldots\lambda_s}(\tilde{A}) \quad (20)$$

and therefore

$$\tilde{\mathcal{L}}(A) = \mathcal{L}(\tilde{A}). \quad (21)$$

But now, with the duality transformation (13), we shall have the *additional property* under the inverse transformation $T^{-1} = T$ (13)

$$G_{\mu\nu,\lambda_1\ldots\lambda_s}(A) = \tilde{G}_{\mu\nu,\lambda_1\ldots\lambda_s}(\tilde{A}) \quad (22)$$

which is easy to check using (19) and the definition of $G_{\mu\nu,\lambda_1\ldots\lambda_s}(A)$ [34, 35, 36]. Therefore we have now the desired property that

$$\mathcal{L}(A) = \tilde{\mathcal{L}}(\tilde{A}). \quad (23)$$

This solves the problem posed in the introduction. In the next two sections we shall discuss some additional properties of this duality map and the complementary gauge transformations.

4 Duality Transformation of High Forms

As we have seen above the duality transformation (12), (13), (14) for the rank-2 tensors ($s=1$) coincides with the *ordinary transposition* of the matrices, which has the property that it squares to $\mathbb{1}$. Our duality operator T , defined above (12), (13), (14) also has the same property $T^2 = \mathbb{1}$ and can be considered therefore as a natural generalization of the transposition operation to the higher-dimensional tensors. This observation allows to define symmetric and antisymmetric tensor gauge fields. Indeed as for the ordinary transposition, we can use the generalized transposition, to define symmetric and antisymmetric parts of a higher-rank tensor gauge field A as

$$A^{\text{sym}} = \frac{1}{2}(A + \tilde{A}), \quad A^{\text{asym}} = \frac{1}{2}(A - \tilde{A}) \quad .$$

They are symmetric or antisymmetric with respect to the generalized transposition. Their explicit forms are

$$\begin{aligned} (A^{\text{sym}})_{\mu\lambda_1\ldots\lambda_s} &= \frac{1}{s+1}(A_{\mu\lambda_1\ldots\lambda_s} + A_{\lambda_1\mu\lambda_2\ldots\lambda_s} + \ldots + A_{\lambda_s\lambda_1\ldots\lambda_{s-1}\mu}) \\ (A^{\text{asym}})_{\mu\lambda_1\ldots\lambda_s} &= \frac{s}{s+1}A_{\mu\lambda_1\ldots\lambda_s} - \frac{1}{s+1}(A_{\lambda_1\mu\lambda_2\ldots\lambda_s} + \ldots + A_{\lambda_s\lambda_1\ldots\lambda_{s-1}\mu}) \end{aligned}$$

The symmetric part A^{sym} thus indeed coincides with the total symmetrization of all indices. The total antisymmetrization would of course vanish, as the gauge field is symmetric in all but one index, but A^{asym} is as antisymmetric as it can be.

It is an interesting question if one can generalize this transposition operation to higher-degree forms. The field strength tensor $G_{\mu\nu,\lambda_1\ldots\lambda_s}$ can be an important example. Let us consider a tensor $G_{\mu_1\ldots\mu_n,\lambda_1\ldots\lambda_s}$ which is antisymmetric in its first n indices and is symmetric in the following s indices. The sum $n + s$ is the tensor rank. A natural generalization of the index permutation P is given by

$$(PG)_{\mu_1\ldots\mu_n,\lambda_1\ldots\lambda_s} = \sum_{i=1}^n \sum_{j=1}^s G_{\mu_1\ldots\mu_{i-1}\lambda_j\mu_i\ldots\mu_n,\lambda_1\ldots\lambda_{j-1}\mu_i\lambda_{j+1}\ldots\lambda_s}. \quad (24)$$

Calculating its square one can get

$$P^2 = (s - n)P + sn\mathbb{1} \quad . \quad (25)$$

Again considering a two-parameter family of operators

$$T = aP + b\mathbb{1}$$

one can get a unique linear combination of P and $\mathbb{1}$ which squares to $\mathbb{1}$. It has the form

$$T = \frac{2}{s+n}P - \frac{s-n}{s+n}\mathbb{1} \quad . \quad (26)$$

For the field strength tensors of the lower rank we shall have

$$\begin{aligned} G_{\mu\nu,\lambda_1}^T &= \frac{2}{3}G_{\mu\lambda_1,\nu} + \frac{2}{3}G_{\lambda_1\nu,\mu} + \frac{1}{3}G_{\mu\nu,\lambda_1} \\ G_{\mu\nu,\lambda_1\lambda_2}^T &= \frac{1}{2}G_{\mu\lambda_1,\nu\lambda_2} + \frac{1}{2}G_{\lambda_1\nu,\mu\lambda_2} + \frac{1}{2}G_{\mu\lambda_2,\lambda_1\nu} + \frac{1}{2}G_{\lambda_2\nu,\lambda_1\mu} \\ \dots\dots\dots &\dots\dots\dots \end{aligned} \quad (27)$$

The above transposition law can be used to define symmetric and antisymmetric parts of the field strength tensors, but it is not clear yet, what is the role of the above construction in the generalization of gauge field theory.

5 *Enhanced Gauge Algebra*

It was observed in [34, 36] that for a certain linear combination of Lagrangian forms the rank two gauge field exhibits an enhanced gauge symmetry, where the extended and

the complementary gauge transformations are realized at the same time. It is therefore interesting to check, whether the sum of extended and of the complementary gauge transformations forms a closed algebra. The sum has the form⁵

$$(\delta_\xi + \tilde{\delta}_\eta)A_{\mu\lambda_1} = \nabla_\mu \xi_{\lambda_1} + \nabla_{\lambda_1} \eta_\mu - ig[A_{\mu\lambda_1}, \xi + \eta] \quad (28)$$

and the commutator of two such sums is

$$[\delta_\xi + \tilde{\delta}_\eta, \delta_\psi + \tilde{\delta}_\chi]A_{\mu\lambda_1}$$

and contains, in particular, the commutator of the extended gauge transformation with the complementary one which we can easily compute

$$[\tilde{\delta}_\eta, \delta_\xi]A_{\mu\lambda_1} = -ig\{\nabla_\mu[\eta, \xi_{\lambda_1}] + \nabla_{\lambda_1}[\eta_\mu, \xi] - ig[A_{\mu\lambda_1}, [\eta, \xi]]\}. \quad (29)$$

Now we can check that the algebra is closed

$$\begin{aligned} [\delta_\xi + \tilde{\delta}_\eta, \delta_\psi + \tilde{\delta}_\chi]A_{\mu\lambda_1} &= ([\delta_\xi, \delta_\psi] + [\tilde{\delta}_\eta, \tilde{\delta}_\chi] + [\delta_\xi, \tilde{\delta}_\chi] + [\tilde{\delta}_\eta, \delta_\psi])A_{\mu\lambda_1} = \\ &= -ig\{\nabla_\mu([\xi, \psi_{\lambda_1}] + [\xi_{\lambda_1}, \psi] + [\eta, \psi_{\lambda_1}] + [\xi_{\lambda_1}, \chi]) + \\ &+ \nabla_{\lambda_1}([\eta, \chi_\mu] + [\eta_\mu, \chi] + [\eta_\mu, \psi] + [\xi, \chi_\mu]) - \\ &- ig[A_{\mu\lambda_1}, ([\xi, \psi] + [\eta, \chi] + [\eta, \psi] + [\xi, \chi])\} \\ &= -ig\{\nabla_\mu \zeta_{\lambda_1} + \nabla_{\lambda_1} \omega_\mu - ig[A_{\mu\lambda_1}, \varphi]\}. \end{aligned} \quad (30)$$

Thus indeed it is a similar transformation with the gauge parameters

$$\begin{aligned} \zeta_{\lambda_1} &= [\xi, \psi_{\lambda_1}] + [\xi_{\lambda_1}, \psi] + [\eta, \psi_{\lambda_1}] + [\xi_{\lambda_1}, \chi] = [\xi + \eta, \psi_{\lambda_1}] + [\xi_{\lambda_1}, \psi + \chi] \\ \omega_\mu &= [\eta, \chi_\mu] + [\eta_\mu, \chi] + [\eta_\mu, \psi] + [\xi, \chi_\mu] = [\xi + \eta, \chi_\mu] + [\eta_\mu, \psi + \chi] \\ \varphi &= [\xi, \psi] + [\eta, \chi] + [\eta, \psi] + [\xi, \chi] = [\xi + \eta, \psi + \chi] \end{aligned} \quad (31)$$

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⁵The definition is $\nabla_\mu \xi_\lambda = \partial_\mu \xi_\lambda - ig[A_\mu, \xi_\lambda]$.

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